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An extended Poisson test for detecting the difference between the past and future rates of extremes of sea wave heights

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Abstract

The occurrence rates of extremes of the natural forces, wave heights and sea levels, are one of the important factors in the design of coastal structures. Especially detecting the difference between the past and future rates is becoming recently an issue remarked in the discussions of climate change adaptation. However the difference is often so faint that the significance has not been examined appropriately nor even discussed in many previous studies of statistical methods of extreme value analysis. One might wonder if the significance is well shown in selecting the non-stationary model in comparison with the stationary one. However the difference of those models should be considered to be checked in the region where the frequently observed values are concentrated, while it will not always hold true in the outer region where the observed values are sparse. It is the same point of arguments as existing the limitation of extrapolation even if the parameters of distribution function are estimated. In the previous research, a criterion of possible extrapolation is given by the degree of experience for one sample. This research discusses to compare two populations of sea extremes by extending the Poisson test for two-sample.

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Keywords: Poisson test, climate change, extreme value statistics,degree of experience, return period

1. Introduction

In design of coastal structures we are very concern the occurrence rate of sea extremes of natural forces, sea levels and wave heights, etc. Especially the difference between the past and future rates of these extremes becomes one

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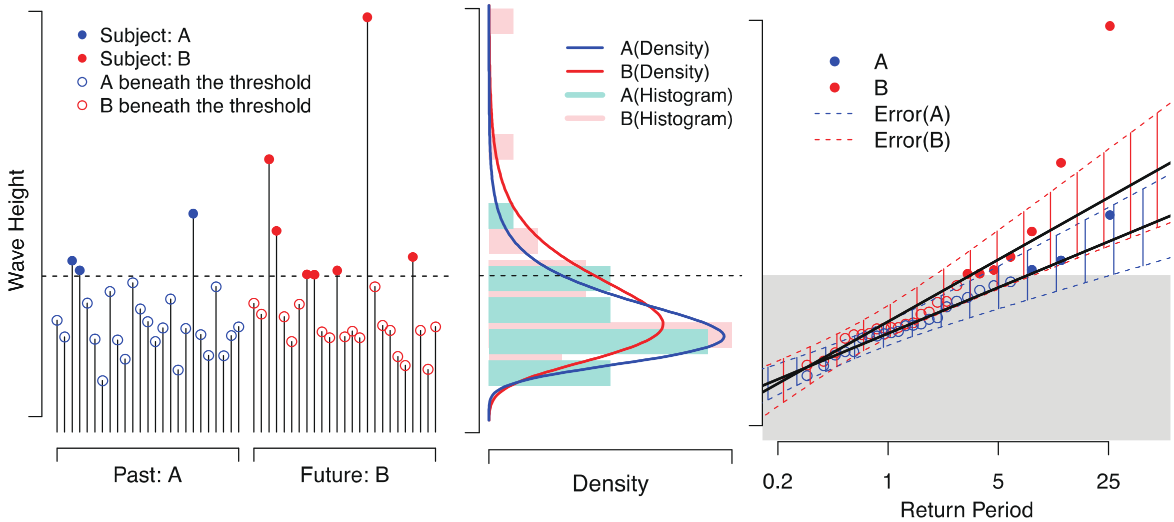
of the important issues in the discussion of climate change. The difference will be often so faint generally that the usual statistical methods in the extreme value analysis can be considered beyond of scope, and even though the resultant differences have been displayed casually, for detecttion the difference has not be enough examined in point of view of statistical test of significance.

For illustration, a dataset of two-sample shown in Fig.1 is used. The numbers of events exceeding the threshold (given by the broken line in Fig. 1) are 3 and 7 in the past and future time of 25 years, which is used for the Poisson test, while the extreme value analysis employ the whole annual maxima including the data beneath the threshold. Detecting the difference of occurrence rates is in other words examining the difference of the tail densities, or distinguishing the return levels whose confidence intervals (or errors) are overlapped, as seen in Fig.1 c).



a) Time series b) Histogram and density c) Return period

Fig. 1. Two-sample of extremes.

Poisson test is generally applied to comparing the rates for two groups. It is easy and convenient to use the Poisson test when the sample size is enough large, and it will be so simple and clear to explain the results even for the policy makers. A new method is proposed here by employing the extrapolation of information. Extrapolation is one of the fundamental properties of extreme value theory. We bring the degree of experience into the Poisson test to make the enhanced version, whose technique is shown step by step.

Nomenclature

F_1	Gumbel distribution of annual maxima
k	occurrence number in a time interval of N years
k_A	occurrence number in the past time interval of N_A years
k_B	occurrence number in the future time interval of N_B years
K	degree of experience for the target level y
K_A	degree of experience for the target level y calculated on the information matrix by the past time sample
K_B	degree of experience for the target level y calculated on the information matrix by the future time sample
I	information matrix derived from the likelihood
R	return period, defined by occurrence rate
R_1	return period, defined by annual maximum distribution

V	variance
y^*	target level for exceedance
λ	occurrence rate, or the function associated with extreme value y and the parameters μ and σ
$\lambda_{1,A}$	annual occurrence rate in the past time
$\lambda_{1,B}$	annual occurrence rate in the future time
μ_1	location parameter included in Gumbel distribution of annual maxima
σ_1	scale parameter included in Gumbel distribution of annual maxima

2. Poisson test for occurrence rates of two-sample

Poisson test is based on a simple idea. It is treated in the introductory textbooks, for example, Hald, 1952. Supposed that k_A is the occurrence number in the past time interval of N_A years and k_B is the occurrence number in the future time interval of N_B years, we test statistically if the number k_A is too small to be occasionally happened (or the number is too large to be happened by chance). The probability of chance is given by a binomial distribution, like as a coin toss model, whose heads and tails correspond to past and future, and whose total number of coin tosses is the merged number $k_A + k_B$.

The probability that the number of occurrence in the past time interval is less than the actual number k_A is given by the following:

$$P(X_A \le k_A \mid X = k_A + k_B) = \sum_{X_A=0}^{k_A} \binom{k_A + k_B}{X_A} (1-q)^{X-X_A} q^{X_A} \tag{1}$$

with the individual probability

$$q = \frac{N_A}{N_B} \frac{\;_{1,A}}{\;_{1,B}} \Bigg/ \left(1 + \frac{N_A}{N_B} \frac{\;_{1,A}}{\;_{1,B}} \right) \tag{2}$$

which is the probability that an individual occurrence belongs to ones of the past time. $\lambda_{1,A}$ and $\lambda_{1,B}$ are the annual occurrence rates in the past and future time, respectively. Thus, $N_A \lambda_{1,A}$ and $N_B \lambda_{1,B}$ are the expected number of occurrences in the past and future time intervals, and they will be close but not the same to the actual numbers of occurrences k_A and k_B. Evaluating the probability of Eq.(1), we employ the null hypothesis:

$$\frac{\;_{1,A}}{\;_{1,B}} = 1 \tag{3}$$

which means that the value of annual occurrence rate $\lambda_{1,B}$ in the future time doesn't change from the one $\lambda_{1,A}$ in the past time. The alternative hypothesis we take here is $\lambda_{1,A}/\lambda_{1,B} < 1$.

The value of the probability given by Eq.(1), is called p-value, and indicates how rare the chance is for the actual state under the null hypothesis. The p-value is equivalently obtained by the following probability:

$$P(X_B \ge k_B \mid X = k_A + k_B) = \sum_{X_B=k_B}^{X} \binom{k_A + k_B}{X_B} (1-q)^{X_B} q^{X-X_B} \tag{4}$$

which is given for the actual occurrence number k_B. It is just by changing the viewpoints.

Fig. 2 shows the contourlines of the p-value given by Eq.(1) or (3) when the null hypothesis of Eq.(3). The p-value is about 0.05 for the case: $k_A = 10$ and $k_B = 20$ and the case: $k_A = 2$ and $k_B = 8$. Only if the sample size is enough large $k_A + k_B = 30$ or if the ratio of occurrence rates is enough large $k_B/k_A = 4$, the null hypothesis can be rejected as the significance of difference. For the moderate ratio of occurrence rates $k_B/k_A = 1.5$ (=33/22), the p-value is 0.1, barely rejectable. So, we understand that it is not easy to detect the difference of occurrences more or less than 5 times for samples from the short interval of 25 years.

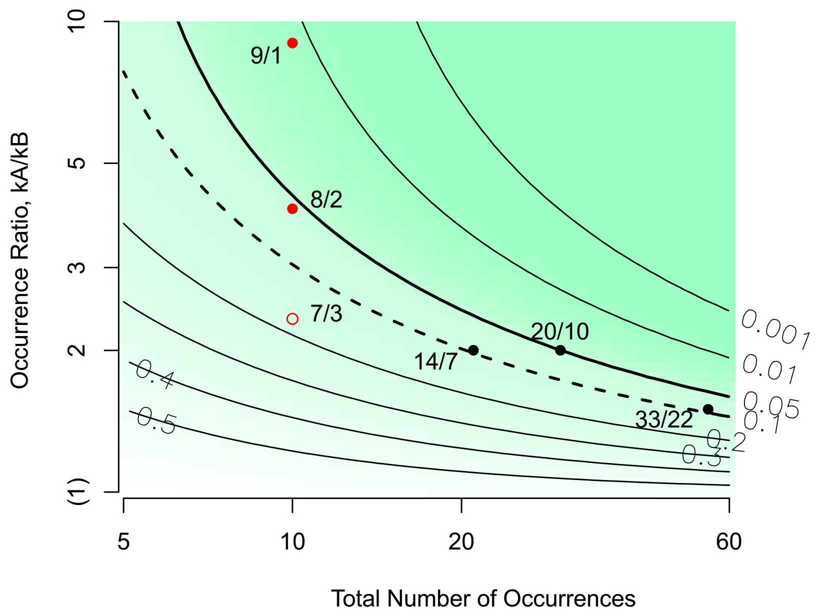


Fig. 2. Contour plot of the *p*-value for Poisson test.

3. Occurrence rates and the degree of experience

3.1. Summary for degree of experience

Generally when the occurrence number of events is k over the time interval of N years, it is natural to estimate the occurrence rate as the following:

$$\hat{\ } = \frac{k}{N} \tag{5}$$

The occurrence number of events k can be decomposed to be expressed as the sum of occurrence numbers k_i for each year, like as $k = k_1 + k_2 + \ldots + k_N$. The occurrence number k_i should be considered to follow a Poisson distributiuon with an occurrence rate λ. As one of the properties of Poisson distribution, the expectation and the variance has the same value as that of λ, and the error variance for the sample mean is inversely proportional to the sample size N, thus for the sample mean expressed by Eq.(5), we obtain

$$V\left(\hat{\ }\right) = \frac{\ }{N} \tag{6}$$

By using the equation above, for the variance of the log transformed quantity, Eq.(6) is expanded as the following:

$$V\left(\log \hat{\ }\right) = \left(\frac{d \log \hat{\ }}{d\ \hat{\ }}\right)^2 V\left(\hat{\ }\right) = \frac{1}{N} \tag{7}$$

Here if the estimated occurrence rate of Eq.(5) is identified to the population's rate λ, the reciprocal value of Eq.(7), that is to say, $N\lambda$ is considered as the total occurrence number k. The degree of experience K, proposed and applied in Kitano et al. (2008), Kitano et al. (2009), Kitano et al. (2010), is defined the reciprocal of the variance of log transformed occurrence rate, which is the left hand side of Eq.(7). Consequently, in this situation of using only counting number of events, the degree of experience K is identical to the occurrence number k. Practically speaking, the degree of experience indicates the occurrence number of events exceeding the target threshold. This is a simple explanation and interpretation for the degree of experience, and it plays a very important role in the Poisson test for two-sample problem of occurrences.

3.2. Degree of experience by using the exceeding and no exceeding data

A Gumbel distribution is employed as the annual maximum distribution F_1 of wave heights, which is given by

$$F_1(y) = \exp\{-\lambda(y;\mu_1,\sigma_1)\}, \qquad \lambda(y;\mu,\sigma) = \exp\left(-\frac{y-\mu}{\sigma}\right) \tag{8}$$

where μ_1 and σ_1 are the location and scale parameters of Gumbel distribution, and F_1 contains the occurrence rate function $\lambda(y;\mu_1,\sigma_1)$ as seen in the second equation of Eq.(8). F_1 is a cumulative distribution, which stands for the non exceedance probability of the target level y, while the probability by Poisson distribution for zero event (of exceeding the level y) is given by

$$\left.\frac{\lambda^k \exp(-\lambda)}{k!}\right|_{k=0} = \exp(-\lambda) \tag{9}$$

It is the reason why the function $\lambda(y)$ is regarded as the occurrence rate of exceeding the target level y. After the relation of the frequency f and period T (= $1/f$), the return period which is the mean period of exceeding the target level y^*, is defined as

$$R = 1/\lambda(y^*;\mu_1,\sigma_1) \tag{10}$$

which is remarked to give slightly different value from that by the ordinary definition of return period as

$$R_1 = \frac{1}{1-F_1(y^*)} \tag{11}$$

Through the occurrence rate function, introduced in Eq.(8), we can define the degree of experience for the target level y^* as the following:

$$\frac{1}{K} = V\{\log\lambda(y^*;\hat\mu_1,\hat\sigma_1)\} = \nabla'\log\lambda(y^*;\hat\mu_1,\hat\sigma_1)\, I^{-1}\nabla\log\lambda(y^*;\hat\mu_1,\hat\sigma_1) \tag{12}$$

where I is the information matrix, and the inverse matrix that takes a place of variance-covariance matrix of parameters. For the evaluation of I, we employ the Hessian matrix in the optimization of likelihood function in the procedure of ML estimation. In the theoretical point of view, the Fisher information matrix is used in place of I, and given by

$$I = \frac{N}{\sigma_1^2}\begin{pmatrix} 1 & \gamma-1 \\ \gamma-1 & \pi^2/6+(\gamma-1)^2 \end{pmatrix} \tag{13}$$

where γ (= 0.5772 …) is Euler's constant.

For Gumbel distribution, the gradient vector is also simply given as

$$\nabla\log\lambda(y^*;\hat\mu_1,\hat\sigma_1) = \frac{1}{\hat\sigma_1}\begin{pmatrix} 1 \\ -\log\lambda(y^*;\hat\mu_1,\hat\sigma_1) \end{pmatrix} \tag{14}$$

Fig.3 shows the degree of experience evaluated by using the Fisher information matrix of Eq.(13) per an annual maximum *K/N*, compared with the exceedance probability *Q* which is the expected number of exceeding the target threshold. It is just confirmed that both values are in the range 0-1 and they decrease as the increase in the return period, and it should be remarked that the degree of experience always surpasses the exceedance probability. It means the profit of extrapolation by the extreme value analysis using the whole annual maxima including non-exceeding cases as well as exceeding cases. Therefore the Poisson test will be expected to work effectively if the degrees of experience are employed in place of the actual numbers of events exceeding the target level.

An additional property of degree of experience should be mentioned here before the applications. Since the degree of experience evaluated by employing the whole data (exceeding and no-exceeding the target level) will take a value bigger than the actual number of occurrences exceeding the level, a new quantity *L*, introduced as the virtual time interval in place of *N* in Eq.(7), should satisfy the following relation parallel to that of Eq.(5).

$$= \frac{K}{L} \tag{15}$$

It is because the degree of experience *K* in Eq.(12) substitutes for k (= $N\,\lambda$) in Eq.(7).

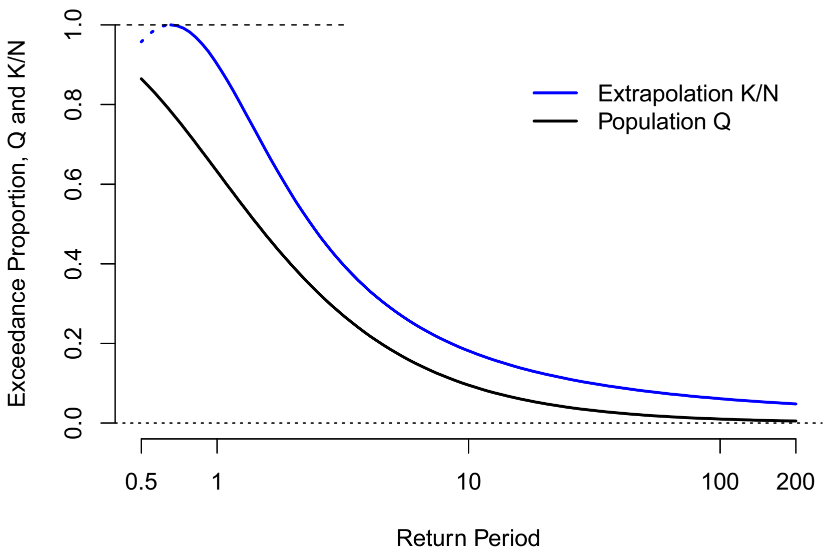


Fig. 3. Degree of experience per 1 year in comparison with exceedance probability.

4. An extended Poisson test

4.1. Proposed manner with the degrees of experience

First of all we fit Gumbel distributions to two samples of the past and future time, and estimate the parameters μ_1 and σ_1 for each sample. Then, by putting the target level y^* in Eq.(8) and (12), we will obtain the degrees of experience K_A and K_B as well as the estimates for occurrence rates $\lambda_{1,A}$ and $\lambda_{1,B}$. Recalling Eq.(15), the virtual time intervals L_A and L_B for the past and future samples, respectively, as

$$L_A = K_A \Big/ \hat{\ }_{1,A}, \qquad L_B = K_B \Big/ \hat{\ }_{1,B} \tag{16}$$

Based on the degrees of experience in the past and future time samples, the probability that the number of occurrence in the time interval of L_A is less than the degree of experience in the past sample K_A, is approximately given by the normal distribution as

$$P(X_A \leq K_A \mid X = K_A + K_B) = \frac{1}{\sqrt{2}} \int_{-\infty}^{Z_A} \exp\left(-\tfrac{z^2}{2}\right) dz \tag{17}$$

where Z_A is the normalized quantity with continuity correction by using the appropriate individual probability.

$$Z_A = \frac{K_A + 0.5 - (K_A + K_B)\,\tilde{q}}{\sqrt{(K_A + K_B)\,\tilde{q}\,(1-\tilde{q})}}, \qquad \tilde{q} = q\,\big|_{N_A/N_B = L_A/L_B,\ \ {}_{1,A}/\ {}_{1,B} = 1} \tag{18}$$

Instead of Eq.(16), it is easy to confirm that the same value is given as the probability that the number occurrence in the time interval of L_B is larger than the degree of experience in the future sample K_B as

$$P(X_B \geq K_B \mid X = K_A + K_B) = \frac{1}{\sqrt{2}} \int_{Z_B}^{\infty} \exp\left(-\tfrac{z^2}{2}\right) dz \tag{19}$$

where Z_B is the normalized quantity with continuity correction.

$$Z_B = \frac{K_B - 0.5 - (K_A + K_B)(1-\tilde{q})}{\sqrt{(K_A + K_B)\,\tilde{q}\,(1-\tilde{q})}} \quad (= -Z_A) \tag{20}$$

4.2. Demonstration by example

For the general purposes of extreme value analysis, ismev package (Heffernan and Stephenson, 2014) and evd package (Stephenson, 2014) are available in the statistical computing environment in R. The estimates of parameters and the Hessian matrices are output straightforwardly for each sample of example in Fig.1. The results by putting them into Eq.(8), (12) and (14), is shown in Table 1.

Table 1. Occurrence rates per year, degrees of experience, virtual time length

Sample	λ (time/year)	K	L (year)
A: the past time	0.13	5.38	41.20
B: the future time	0.30	9.41	31.26

The proposed extended Poisson test can be done with just three values: K_A, K_B, L_A/L_B. Under the null hypothesis of Eq.(4) that the occurrence rates are equivalent, the *p*-value given by Eq.(17) or (19), is 0.092 (< 0.10). So we can reject the null hypothesis for the significant level of 0.1, while the *p*-value is 0.16 (> 0.10) by putting the expected numbers of occurrence 3.3 (=0.13 x 25) and 7.5 (=0.30 x 25) in the ordinary test by Eq.(1) or (4).

Furthermore the type 2 error of failing to detect the change can be obtained as

$$1 - P\left(X_A \leq 5.38 \middle| X = 5.38 + 9.41\right) = 0.40 \tag{21}$$

for the hypothesis of unequivalent rates as

$$\frac{{}_{1,A}}{{}_{1,B}} = \frac{3.3}{7.5} \tag{22}$$

This probability will be not so bad, because the type 2 error takes almost the same value

$$1 - P\left(X_A \le 3.3 \middle| X = 3.3 + 7.5\right) = 0.37 \tag{23}$$

which is, however, remembered to be not so small significance (p-value = 0.16 < 0.10), and it is noted that it takes reasonably the bigger values for more significant cases.

$$1 - P\left(X_A \le 2 \middle| X = 2 + 8\right) = 0.63 \tag{24}$$

$$1 - P\left(X_A \le 1 \middle| X = 1 + 9\right) = 0.86 \tag{25}$$

whose p-values are 0.05 and 0.01, respectively, as seen in Fig.2. High significance might miss the risk, unless the sample size is enough.

It is very interesting that this extended Poisson test can reduce the p-value (type 1 error) as keeping almost the same value of type 2 error. It can be said to be relatively easy to detect the change. The type 2 error is to miss the risk, while the type 1 error is to commit the unnecessary worry. The type 2 error is very important to be prevented from growing for the design of maritime structures to protect the coastal disasters. Unfortunately we cannot diminish the type 2 error drastically, but to avoid increasing the probability is as much as we can. Enhanced Poisson test proposed here in this study is favorable in point of view of balancing these two kinds of errors.

5. Conclusions

To detect the difference of the past and future occurrences of extremes, we proposed to extend the Poisson test for two-sample with the degree of experience. It is effective to balance the two kinds of errors.

The extended Poisson test proposed in this study is also applicable to the samples of sea extremes in POT method, and it will be more effective if the scale and shape parameters are common for two-sample.

Acknowledgements

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Appendix A. Two samples used for illustrations

The dataset used in this study is generated by these command by evd package in R.

```
> library(evd)
> A <- round(0.9 * rgumbel(25, loc=log(3/25)), 2)
> B <- round(1.2 * rgumbel(25, loc=log(7/25)), 2)
```

And the values are also shown below for reproduction or check by the readers.

```
> A
  [1]  3.59  3.05  5.50  5.19  4.14  2.98  1.65  4.51  2.95  2.34  4.79  3.95  3.54
 [14]  2.90  3.34  4.27  1.99  3.32  7.01  3.13  2.46  4.66  2.45  3.09  3.38
> B
  [1]  4.13  3.78  8.75  6.46  3.70  2.90  4.10  5.06  5.05  3.22  3.03  5.18  3.06
 [14]  3.24  3.02 13.31  4.66  3.42  3.26  2.42  2.14  5.62  3.26  2.01  3.38
```

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